Minimal Model Program.

Learning Seminar.

Week 18.

- Asymptotic multiplier ideals.
- Siu's invariance of plurigenera.

Siu's invariance of plurigenera:

Some preliminary results...

Lemma (Kodaira): X a normal complete variety, D big & Cartier on X. Then, for any Cartier divisor M on X, we have

$$H^0(X, \mathcal{O}_X(\ell D - M)) \neq 0$$

for $\ell \gg 0$.

Theorem (Nadel): X smooth proj. L big Weil divisor. A a nef divisor on X. Then

asymptotic mult ideal of mL and $m \geq 1$.

$$H^i(X, \mathcal{O}_X(K_X + mL + A) \otimes \mathcal{J}(\|mL\|)) = 0 \quad \text{for } i > 0.$$

Corollary (global generation): In the setting of the previous theorem. If B is a globally generated line bundle on X. Then, for any $m \geq 1$,

$$\mathcal{O}_X(K_X + nB + A + mL) \otimes \mathcal{J}(m\|L\|)$$

is globally generated, where $n = \dim X$.

B gg $H^i(D - nB) = 0$, then D is g.g.

V smooth proj variety, m-th plurigenera.

$$P_m(V) = \dim H^0(V, \mathcal{O}_V(mK_V)).$$

birational invariant for smooth projective varieties.

It was an open question whether they are constant under deformation

variety of general type $=$ K_V big.

Theorem (Deformation invariance of plurigenera):

smooth proj morphism $\pi: X \longrightarrow T$ of smooth quasi-proj varieties. For $t \in T$, X_t denote the corresponding fiber.
Assume each X_t is irreducible & of general type.
Then, for every $m \geq 1$, the plurigenera

$$P_m(X_t) = h^0(X_t, \mathcal{O}_{X_t}(mK_{X_t}))$$

are independent of $t \in T$.

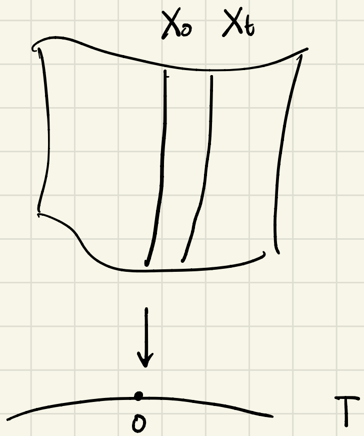
Remarks: 1) We can assume T is a smooth curve.

2) $P_m(X_t)$ are upper semicontinuous in the Zariski topology.

Fix $0 \in T$ and $m \geq 1$, we want to prove

$$P_m(X_0) \leq P_m(X_t)$$

for any t in some neighborhood of $0 \in T$.



$P_m(X_0) \le P_m(X_t)$ holds on some neighborhood of $0 \in T$.

Remark: Assume K_{X_0} is nef & big.

By KV vanishing $H^i(X_0, \mathcal{O}_{X_0}(mK_{X_0})) = 0$, $i > 0$, $m \geq 2$.

By semicont, we conclude $H^i(X_t, \mathcal{O}_{X_t}(mK_{X_t})) = 0$ for t near 0.

Then, for t near 0, $P_m(X_t) = \chi(X_t, \mathcal{O}_{X_t}(mK_{X_t}))$

Thus, the result follows from deformation invariance of χ.

Remark: for some $p > 0$ there exists $D \in |pK_{X_0}|$

so that $\mathcal{J}(X_0, c \cdot D) = 0$, for $c < 1$

Then Nadel vanishing $H^i(X_0, \mathcal{O}_{X_0}(mK_{X_0})) = 0$ for $m \geq p$.

which implies $P_m(X_t)$ is locally constant near zero.

Proof: $\mathcal{P}_m = \pi_* \mathcal{O}_X(mK_X)$ is a v.b on T

$\phi_m(t) : \mathcal{P}_m(t) = \mathcal{P}_m \otimes \mathbb{C}(t) \longrightarrow H^0(X_t, \mathcal{O}_{X_t}(mK_{X_t}))$.

is an isomorphism for t general

We want to show

$\phi_m := \phi_m(0) : \mathcal{P}_m(0) \longrightarrow H^0(X_0, \mathcal{O}_{X_0}(mK_{X_0}))$ is surj.

Any $s \in \Gamma(X_0, \mathcal{O}_{X_0}(mK_{X_0}))$ lifts to

$\tilde{s} \in \Gamma(X, \mathcal{O}_X(mK_X))$.

Fix $m \geq 1$, $k \geq 1$.

$$\Gamma(X, \mathcal{O}_X(mkK_X)) \longrightarrow \Gamma(X_0, \mathcal{O}_{X_0}(mkK_{X_0}))$$

gives a graded linear system of ideals on X_0.

We denote the corresponding multiplier ideal by

$$\mathcal{J}(\|mK_X\|_0) := \mathcal{J}(X_0, \|mK_X\|_0)$$

Lemma:

m- pluricanonical sections.

$$H^0(X_0, \mathcal{O}_{X_0}(mK_{X_0}) \otimes \mathcal{J}(\|(m-1)K_X\|_0)) \subseteq$$

$$\mathrm{Im}\left(H^0(X, \mathcal{O}_X(mK_X)) \longrightarrow H^0(X_0, \mathcal{O}_{X_0}(mK_{X_0}))\right).$$

any m-pluricanonical section that vanishes on

$\mathcal{J}(\|(m-1)K_X\|_0)$ lifts.

Proof: $b = I(X, \mathcal{O}_X(pK_X))$ $\quad p \gg 0$.

$f: V \longrightarrow X$, log resolution of b, strict transform of X_0.

$b \cdot \mathcal{O}_V = \mathcal{O}_V(-F)$, $\quad f^* X_0 = V_0 + \sum' a_i E_i$.

$$B = (K_V + V_0) - f^*(K_X + X_0) - \left[\tfrac{1}{p} F\right].$$

$$= K_{V/X} - \left[\tfrac{1}{p} F\right] - \textstyle\sum' a_i E_i.$$

$$\begin{tikzcd} V_0 \arrow[r, hook] \arrow[d, "f_0"'] & V \arrow[d, "f"'] \arrow[dd, bend left, "h"] \\ X_0 \arrow[r, hook] \arrow[d] & X \arrow[d, "\pi"'] \\ \{0\} \arrow[r, hook] & T \end{tikzcd}$$

$$\mathcal{J}(X_0, \|(m-1)K_X\|_0) = f_{0,*}\mathcal{O}_{V_0}(B|_{V_0})$$

$$h_* \mathcal{O}_V(B + f^*(mK_X)) \subseteq \pi_* \mathcal{O}_X(mK_X).$$

It suffices to show that the map

$$h_*(\mathcal{O}_V(B+f^* mK_X)) \longrightarrow h_*(\mathcal{O}_{V_0}(B+f^*(mK_X))).$$

its sections are m-pluricanonical forms vanishing at $\mathcal{J}(\|(m-1)K_X\|_0)$

is surjective

Observe that

$$f^*(mK_X)+B-V_0 \sim K_{V/T} \underbrace{+f^*((m-1)K_X) - \left[\tfrac{1}{p}F\right]}_{\text{big \& nef over } T} - h^*(0).$$

Then, the relative version of Kawamata-Viehweg vanishing applied to the morphism $V \longrightarrow T$ gives us the above surj $\square$.

Review:

$P_m(X_0) \le P_m(X_t)$

We constructed $\mathcal{J}(\|mK_X\|_0)$ on the central fiber which measures the sing of the restricted m-pluricanonical section. Any m-pluricanonical section on X_0 lying on the subspace

$$\operatorname{im} \phi_m \subseteq H^0(X_0, \mathcal{O}_{X_0}(mK_{X_0})).$$

vanishes along $\mathcal{J}(\|mK_X\|_0)$ by definition.

The Lemma, on the other hand, shows that any section

$s \in H^0(X_0, \mathcal{O}_{X_0}(mK_{X_0}) \otimes \mathcal{J}(\|(m-1)K_X\|_0)$ extends

to $H^0(X, \mathcal{O}_X(mK_X))$.

Aim: Any $s \in \Gamma(X_0, \mathcal{O}_{X_0}(mK_{X_0}))$ vanishes along

$$\mathcal{J}(\|(m-1)K_X\|_0).$$

$\mathcal{J}(\|(m-1)K_X\|_0)$ measures singularities of the restricted m-th pluricanonical section.

$\mathcal{J}(\|mK_{X_0}\|)$ measures singularities of the m-th pluricanonical sections of X_0

$$H^0(X_0, \mathcal{O}_{X_0}(mK_{X_0}) \otimes \mathcal{J}(\|mK_{X_0}\|)) \simeq H^0(X_0, \|mK_{X_0}\|).$$

Remark: If $\mathcal{J}(X_0, \|mK_{X_0}\|) \subseteq \mathcal{J}(X_0, \|(m-1)K_X\|_0)$.

then we win

approximation of this containment.

Main claim: There exists $a \geq 1$, and $D_0 \subseteq X_0$ fixed.

$$\mathcal{J}(X_0, \|kK_{X_0}\|)(-D_0) \subseteq \mathcal{J}(X_0, \|(k+a-1)K_X\|_0).$$

for every $k \geq 1$. Moreover, there exists $s_D \in \Gamma(aK_{X_0})$ vanishes along D_0.

$\mathcal{J}(X_0, \|kK_{X_0}\|)$ $\qquad$ $\mathcal{J}(X_0, \|kK\|_0)$

are not so different.

Proof of claim: B very ample $\qquad$ $n = \dim X$

$$2K_X + (n+1)B \sim G \geq 0.$$

$\mathcal{O}_X(G)$ is free

By Kodaira's Lemma. $\quad aK_X - G \sim D \geq 0.$

We can assume D & G contains no fibers.

By construction $D + G \sim aK_X$.

determines $s_D \in \Gamma(X, \mathcal{O}_X(aK_X))$. vanishes at D

and does not vanish on any fiber.

$$D_0 = D|_{X_0} \quad \text{and} \quad B_0 = B|_{X_0}.$$

We will prove

$$\mathcal{J}(\|kK_{X_0}\|)(-D_0) \subseteq \mathcal{J}(\|(k+a-1)K_X\|_0) \quad \text{by ind on } k. \quad \star$$

For $k=1$, it suffices to show $\mathcal{O}_{X_0}(-D_0) \subseteq \mathcal{J}(\|aK_X\|_0)$

By construction, $\underline{aK_X - D \sim G}$ is free

$N \in |2G|$ general enough, we have

$$\underset{\substack{\shortparallel \\ \mathcal{O}_{X_0}(-D_0)}}{\mathcal{J}(X_0, D_0)} = \mathcal{J}(X_0, \tfrac{1}{2}(2D+N)|_{X_0}) \subseteq \mathcal{J}(X_0, \|aK_X\|_0).$$

Assume ($\star$) for k and prove

$$\mathcal{O}_{X_0}((k+a)K_{X_0} - D_0) \otimes \mathcal{J}(\|(k+1)K_{X_0}\|) \subseteq \quad (\star\star)$$

$$\mathcal{O}_{X_0}((k+a)K_{X_0}) \otimes \mathcal{J}(\|(k+a)K_X\|_0).$$

is globally generated.

By g.g, it suffices to check ($\star\star$) on global section.

$(k+a)$- canonical forms on X_0 vanishing along $\mathcal{J}(\|(k+1)K_{X_0}\|) \otimes \mathcal{O}_{X_0}(-D_0)$

necessarily vanish along $\mathcal{J}(\|(k+a)K_X\|_0)$.

we want

Using $(\star)$, we have the following containments.

$$\mathcal{J}(\|(k+1)K_{X_0}\|)\otimes \mathcal{O}_{X_0}(-D_0) \subseteq \mathcal{J}(\|kK_{X_0}\|)\otimes \mathcal{O}_{X_0}(-D_0)$$

$$\overset{\star}{\subseteq} \mathcal{J}(\|(k+a-1)K_X\|_0)$$

if $s \in \Gamma(X_0, \mathcal{O}_{X_0}((k+a)K_{X_0}))$ vanishes along

$$\mathcal{J}(\|(k+1)K_{X_0}\|)\otimes \mathcal{O}_{X_0}(-D_0)$$

then it vanishes along $\mathcal{J}(\|(k+a-1)K_X\|_0)$, then it lifts so it lies in $\operatorname{Im}\phi_{k+a} \subseteq H^0(\mathcal{O}_{X_0}((k+a)K_{X_0}))$

then it vanishes along $\mathcal{J}(\|(k+a)K_X\|_0)$.

$$\Gamma(\mathcal{O}_{X_0}((k+a)K_{X_0}-D_0)\otimes \mathcal{J}(\|(k+1)K_{X_0}\|)) \subseteq$$

$$\Gamma(\mathcal{O}_{X_0}((k+a)K_{X_0})\otimes \mathcal{J}(\|(k+a)K_X\|_0))$$

$$\Downarrow$$

$$\mathcal{O}_{X_0}((k+a)K_{X_0}-D_0)\otimes \mathcal{J}(\|(k+1)K_{X_0}\|) \subseteq$$

$$\mathcal{O}_{X_0}((k+a)K_{X_0})\otimes \mathcal{J}(\|(k+a)K_X\|_0).$$

$\square$

We return to the proof of invariance of plurigenera...

Main claim: For any $k \geq 1$.

$$\mathcal{J}(X_0, \|kK_{X_0}\|)(-D_0) \subseteq \mathcal{J}(X_0, \|(k+a-1)K_X\|_0)$$

$a \geq 1$, $s_D \in \Gamma(X_0, \mathcal{O}_{X_0}(aK_{X_0}))$ and $D_0 \subseteq X_0$ are fixed

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$$s \in \Gamma(X_0, \mathcal{O}_{X_0}(mK_{X_0}))$$

we want to prove that s vanishes along

$$\mathcal{J}(X_0, \|(m-1)K_X\|_0)$$

(then, the first lemma implies that it lifts).

$$s^l \cdot s_D \in \Gamma(X_0, \mathcal{O}_{X_0}((ml+a)K_{X_0}))$$

s^l vanishes along $\mathcal{J}(\|mlK_{X_0}\|)$

s_D vanishes along D_0.

$s^l s_D$ vanishes along $\mathcal{J}(\|mlK_{X_0}\|)(-D_0) \subseteq \mathcal{O}_{X_0}$

$s^l \cdot s_D \in \Gamma(X_0, \mathcal{O}_{X_0}(ml+a)K_{X_0}))$

s^l vanishes along $\mathcal{J}(\|mlK_{X_0}\|)$

s_D vanishes along D_0.

$s^l s_D$ vanishes along $\mathcal{J}(\|mlK_{X_0}\|)(-D_0) \subseteq \mathcal{O}_{X_0}$

$$\cap\!\mid \quad \text{Main claim}$$

$$\mathcal{J}(\|mlK_X\|_0)$$

$$\cap\!\mid \quad \text{subadditivity.}$$

$$\mathcal{J}(\|mK_X\|_0)^l.$$

$s^l s_D \in \Gamma(X_0, \mathcal{O}_{X_0}((ml+a)K_{X_0}) \otimes \mathcal{J}(\|mK_X\|_0)^l)$, $l \gg 0$

Then s vanish along $\overline{\mathcal{J}(\|mK_X\|_0)} = \mathcal{J}(\|mK_X\|_0)$.

$$\mathcal{J}(\|mK_X\|_0) \subseteq \mathcal{J}(\|(m-1)K_X\|_0).$$

s vanishes

Then s lifts. $\square$.